The space of left-orderings of a solvable group with finite Prüfer rank

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Abstract

Left-orderable countable groups with finitely many left-orderings have been classified by Tararin and turn out to be a special class of finite-rank-solvable groups. Here, we complete the picture for finite-rank-solvable groups, showing that their space of left-orderings is either finite or homomorphic to the Cantor set.

1 Introduction

The space of left-orderings, $\mathcal{LO}(G)$, of a left-orderable group G, is the set of all possible left-orderings on G endowed with a natural topology that makes it compact, Hausdorff and totally disconnected, see [20] or §2.1. It was proved by Linnell that this space is either finite or uncountable [8].

1.1 Isolated ordering

The problem of relating the topology of $\mathcal{LO}(G)$ with the algebraic structure of G has been of increasing interest since the discovery by Dubrovina and Dubrovin that the space of left-orderings of the braid groups is infinite and yet contains isolated points [3]. Recently, more examples of groups showing these two behaviors have appeared in the literature [2, 5, 6, 11]. All this groups contain free subgroups (however, non-trivial free products of groups have no isolated left-orderings [18]).

Recall that an Abelian group A has finite (Prüfer) rank at most n if A embeds in a quotient of $\mathbb{Q}^n$. A finite-rank-solvable group Γ is a group admitting a normal filtration $\{1\} = \Gamma_0 \lhd \Gamma_1 \lhd \ldots \lhd \Gamma_n = \Gamma$ such that every (Abelian) quotient has finite rank.

Theorem 1.1. *The space of left-orderings of a virtually finite-rank-solvable group is either finite or without isolated points (i.e. homeomorphic to the Cantor set).*

Groups admitting only finitely many left-orderings (a *Tararin* group for short) have been classified by Tararin [7][Theorem 5.2.1]. These groups turn out to be finite-rank-solvable, and the virtually polycyclic ones are virtually nilpotent (see §2.2). In particular, we deduce

Corollary 1.2. *The space of left-ordering of a left-orderable, virtually polycyclic group of exponential growth has no isolated points. In particular, it is homeomorphic to a Cantor set.*

1.2 Conradian versus non-Conradian orderings

The dichotomy shown in Theorem 1.1 reminds of a similar one, this time for all groups but in restriction to Conradian orderings. Indeed, in [16] the first author proved that the space of Conradian orderings of a countable group is either finite or homeomorphic to the Cantor set. In particular

this implies the general dichotomy for left-orderable groups having only Conradian orderings, such as groups of sub-exponential growth [9, 10].

On the other hand, in [17], Theorem 1.1 was proved for a special class of finite-rank-solvable groups, which happens to be the class of all left-orderable groups with only finitely many Conradian left orderings. This class is very similar (but larger) than the class of Tararin groups (see §2.2): indeed it consists of all groups admitting a normal filtration $\{1\} = \Gamma_0 \lhd \Gamma_1 \lhd \ldots \lhd \Gamma_n = \Gamma$ such that every quotient Γ_i/Γ_{i-1} for $i \geq 1$ is isomorphic to a subgroup of $\mathbb{Q}$, and such that Γ_i/Γ_{i-2} is non-Abelian whenever $i \geq 2$.

1.3 Ingredients of the proof

The proof of Theorem 1.1 has several ingredients. One is the well developed theory of Conradian ordering started by Conrad in [1], and pursued by other autors such as [9, 15, 10, 16] among others (see [10] for more references). These are left-orderings satisfying the additional property that $f \succ id$ and $g \succ id$ implies $fg^2 \succ g$. A second ingredient is the well known fact that countable left-orderable groups acts faithfully on the real line [4]. This allows us to use the dynamical conterpart of Conradian orderings revealed by Navas in [10], and also the machinery developed by Plante of invariant and quasi-invariant measures for solvable groups acting on the real line [13]. Finally, there are two notions involving finite-rank-solvable groups on which the proof strongly relies. One is the nilpotent radical of a finite-rank-solvable group Γ, which is the maximal, normal nilpotent subgroup of Γ. This subgroup is *large* inside Γ, meaning that there is always a normal, finite index subgroup $\widetilde{\Gamma}$ such that $[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is nilpotent [19]. The second notion, is the Hirsch number of a virtually finite-rank-solvable group, which is defined as follows: Let H be a finite index solvable subgroup of G, and let A_i be i'th Abelian quotient associated to some normal filtration of H. The Hirsh number of G is then defined to be the sum of the $\mathbb{Q}$-dimensions of the $A_i \otimes \mathbb{Q}$. This number is easily seen to be independent of the solvable subgroup H, and of the normal filtration [19]. Moreover it is finite for groups of finite rank, non-increasing when passing to a subgroup, and additive under short exact sequences. This implies in particular that if $H \lhd H' < \Gamma$ such that $\mathrm{rank}(H'/H) > 0$, then $\mathrm{rank}(H) < \mathrm{rank}(\Gamma)$. On the other hand if H is normal in Γ and $\mathrm{rank}(H) > 0$, then $\mathrm{rank}(\Gamma/H) < \mathrm{rank}(\Gamma)$. We shall use these remarks freely in the sequel.

2 Preliminaries

2.1 The topology on $\mathcal{LO}(G)$

A basis of neighborhoods in $\mathcal{LO}(G)$ is the family of the sets $V_{f_1,\ldots,f_k} := \{\preceq \mid id \preceq f_1, \ldots, id \preceq f_k\}$, where $\{f_1, \ldots, f_k\}$ runs over all finite subsets of G. If G is countable, then this topology is metrizable. For instance, if G is finitely generated, we may define $dist(\preceq, \preceq') = 1/2^n$, where n is the first integer such that $\preceq$ and $\preceq'$ do not coincide on n-th ball (with respect to some generator system).

The following definition is classical. Given a left-ordered group $(G, \preceq)$ and a subgroup H, we say that H is *convex* if for every $g \in G$ such that $h_1 \preceq g \preceq h_2$, for $h_1,\ h_2$ in H, we have that $g \in H$. Convex subgroups have the nice property that they induce a total ordering on the left-cosets G/H by

$$g_1H \preceq^* g_2H \Leftrightarrow g_1h_1 \preceq g_2h_2 \text{ for all } h_1, h_2 \text{ in } H.$$

This ordering is invariant by the left translation action of G on G/H (so in particular, if H is also normal, we have that G/H is a left-orderable group). It follows that $\preceq$ can be decompose as the

order on the H-cosets and the order restricted to H. More precisely we have that

$$id \prec g \Leftrightarrow \left\{ \begin{array}{c} H \prec^* gH, \text{ or} \\ H = gH \text{ and } id \prec g \end{array} \right.$$

Elaborating on this, we conclude (see [17] for more details)

Proposition 2.1. *Let $\preceq$ be a left-ordering on G and let H be a convex subgroup. Then there is a continuous injection $\mathcal{LO}(H) \to \mathcal{LO}(G)$, having $\preceq$ in its image. Moreover, if in addition H is normal, then there is a continuous injection $\mathcal{LO}(H) \times \mathcal{LO}(G/H) \to \mathcal{LO}(G)$ having $\preceq$ in its image.*

Remark 2.2. Let $\preceq$ be a left-ordering on G, and H a normal convex subgroup. Then, it is not hard to check that if the restriction of $\preceq$ to H and the projection of $\preceq$ to G/H are Conradian, then $\preceq$ is also Conradian.

2.2 Tararin groups

We give a slightly modification of the original statement of Tararin [7, Theorem 5.2.1], which describe groups admitting only finitely many left-orderings. For the statmenet recall that a series

$$\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G,$$

is said to be *rational* if each quotient G_{i+1}/G_i is torsion-free rank-one Abelian.

Theorem 2.3 (Tararin)**.** *Let G be a left-orderable group. If G admits only finitely many left-orderings, then G admits a unique (hence normal) rational series*

$$\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G,$$

such that, for every $2 \leq i \leq n$, in the conjugation action of G_i/G_{i-1} on G_{i-1}/G_{i-2}, there is an elements acting as a multiplication by a negative rational number. Conversely, if G admits such a rational series, then the number of left-orderings on G is 2^n.

Remark 2.4. The left-orderings on a group G fitting in Theorem 2.3 are very easy to describe. Indeed, if $\{1\} = G_0 \lhd \ldots \lhd G_n = G$, is the associated rational series, then on each quotient G_i/G_{i-1}, being rank-one Abelian, there are only two possible left-orderings. For every i, let $\preceq_i$ be a choice of an ordering on G_i/G_{i-1}. Then we can produce a left-orderings on G by declaring

$$g \succ id \Leftrightarrow \left\{ \begin{array}{l} gG_{n-1} \succ_n G_{n-1} \text{ , or} \\ g \in G_{n-1} \text{ , and } g \succ_{n-1} id \text{, or} \\ \qquad\quad \vdots \\ g \in G_1 \text{ , and } g \succ_1 id. \end{array} \right.$$

Clearly, in this way we can produce all the 2^n possible left-orderings (in fact, it is easy to show that they are all Conradian). Moreover, in any such ordering, the groups G_i are convex.

Corollary 2.5. *Let G be a virtually polycyclic group admitting only finitely many left-orderings. Then it admits a unique filtration such that $G_i/G_{i-1} \simeq \mathbb{Z}$. The action of (the generator of) G_i/G_{i-1} on G_{i-1}/G_{i-2} is by multiplication by -1.*

Since in a virtually polycyclic group, the group generated by $\{g^2 \mid g \in G\}$ has finite index [14], we conclude

Corollary 2.6. *A virtually polycyclic group having only finitely many left-orderings is virtually nilpotent.*

2.3 Dynamical realization and Conradian orderings

As mentioned in the Introduction, one main ingredient in our proof of Theorem 1.1 is the fact that countable left-orderable groups acts naturally by order preserving automorphism of the real line, and vice versa, a group acting faithfully by order preserving automorphism of the real line is left-orderable [4].

More precisely, given a left-ordered group $(G, \preceq)$, there is an embedding of G into $Homeo_+(\mathbb{R})$, the group of order preserving automorphism of the real line, such that:

- G acts without global fixed points, and
- for f, g in G, we have that $f \prec g \Leftrightarrow f(0) < g(0)$.

We call such an action, a *dynamical realization* of $(G, \preceq)$. Conversely, given an embedding of G into $Homeo_+(\mathbb{R})$, we can induce a left- ordering as follows: Take $(x_1, x_2, \ldots)$ a dense sequence in $\mathbb{R}$, and declare that an element $g \succ id$ if and only if $g(x_i) > x_i$, where i is such that $g(x_j) = x_j$ for every $j < i$. We call such an ordering, an *induced ordering* from the action. Note that with this procedure que can recover a left-ordering from its dynamical realization by taking $x_1 = 0$.

In the special case that the ordering on G is Conradian (*e.g.* when G is nilpotent) we can deduce many more properties of this action

Theorem 2.7 (Navas [10])**.** *Consider the dynamical realization of a Conradian ordering of G, and let $I_g \subseteq \mathbb{R}$ be an open interval fixed by $g \in G$. Then, for any $f \in G$, either $f(I_g) = I_g$ or $f(I_g) \cap I_g = \emptyset$.*

The above theorem implies, for instance, that in a dynamical realization of a Conradian ordering, the set of elements having fixed point is a normal subgroup. From this, together with Hölder's Theorem stating that every group acting freely on the real line is Abelian (see for instance [4]), one can easily deduce (compare with [1])

Corollary 2.8. [10] *Let G be a countable group, $\preceq$ be a Conradian ordering of G, and $N \subset G$ be the set of elements having a fixed point in the dynamical realization of $(G, \preceq)$. Then N is a normal subgroup of G. Moreover, if there is $g \in G$ having no fixed point (for instance if G is finitely generated), then G/N is a non-trivial torsion-free Abelian group which acts freely on the (non-empty) set of global fixed points of N.*

3 Proof of the main Theorem

Let Γ be a virtually finite-rank-solvable group. We let $\widetilde{\Gamma}$ be a normal, finite index subgroup of Γ such that $[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is nilpotent. We also let R be the nilpotent radical of $\widetilde{\Gamma}$. It is known that R is a characteristic subgroup of $\widetilde{\Gamma}$, hence it is normal in Γ (to all of this see [19]). Finally, we let $\preceq$ be a left-ordering on Γ.

Theorem 1.1 follows immediately if R has finite index in $\widetilde{\Gamma}$ (as it is the case if the Hirsch number of Γ is 1). Indeed, in this case Γ is virtually nilpotent, so it admits only Conradian orderings [9]. Moreover, a Conradian ordering is isolated only if the group is a Tararin group [10]. In what follows, we assume that $\widetilde{\Gamma}/R$ is infinite, and that Theorem 1.1 follows for every virtually finite-rank-solvable group having smaller Hirsch number than Γ.

We shall start by proving the Theorem 1.1 in the special case when the ordering $\preceq$ on Γ is realized by some affine action on $\mathbb{R}$.

Lemma 3.1. *Suppose the ordering $\preceq$ on Γ is induced from a faithful (order preserving) affine action on the real line. Then if Γ is non-Abelian, $\preceq$ is non-isolated.*

Proof: By assumption, Γ has both non-trivial homotheties and non-trivial translations. It follows that the countable subset Ω consisting of the point in $\mathbb{R}$ which are fixed by some non-trivial element of Γ, is dense in $\mathbb{R}$. Therefore, if $\preceq$ is the left-ordering on Γ induced from the dense sequence $(x_1, x_2, \dots)$ (see §2.3), and if $x \notin \Omega \cup \{x_1\}$ (so x has free orbit), then the left-ordering induced from x will be different from $\preceq$ (because there is a non-trivial homothecy having its unique fixed point between x and x_1). On the other hand if x is close enough to x_1, $\preceq$ will be unchanged on any fixed finite subset of Γ. In other words, $\preceq$ is not isolated. $\square$

The general case, lies somewhere between the two previously described cases. We will focus on the dynamics of the nilpotent subgroup R and on its subgroup of elements having fixed points. We have

Lemma 3.2. *Let N be torsion-free nilpotent group with finite rank acting by increasing homeomorphisms on $\mathbb{R}$. If all its elements have fixed points, then N has a global fixed point.*

Proof: Note that the statement is easy and well-known for finitely generated Abelian groups (see for instance Corollary 2.8). We will argue by induction on the length of nilpotency. If the length is 0, the group is trivial so there is nothing to prove. Let Z be the center of N. This is a finite rank Abelian group, hence it contains a subgroup Z' isomorphic to $\mathbb{Z}^d$ such that Z/Z' is torsion. It follows that the (closed) set A of fixed points of Z' is non-empty. Since Z/Z' is torsion, A is also the set of fixed points of Z. It follows from Theorem 2.7, that the complement of A is a disjoint union $\bigsqcup_i I_i$ of open intervals permuted by N. Moreover, since every element of N has fixed points, again from Theorem 2.7, we have that every element in N must fix some $a \in A$. Extending in a piecewise affine way the action of N on A, we obtain a new action of N on $\mathbb{R}$ by increasing homeomorphisms which factors through N/Z. Moreover, this new action coincides with the initial one on the set A. Thus, this action is such that every element has fixed points in A. So, by induction, N/Z has a global fixed point (in A), from which the lemma follows. $\square$

Corollary 3.3. *Let G be a finite-rank nilpotent group acting by increasing homeomorphisms on $\mathbb{R}$. Then G has a symmetric generating system $\mathcal{G}$ such that for some $x \in \mathbb{R}$, $\mathcal{G}(x)$ is bounded (in the language of [13], this says that G is* boundedly generated*).*

Proof: The proof is obvious if G has a global fixed point, so we assume the contrary. Let $N \subseteq G$ be the set of elements having a fixed point. Then, from Lemma 3.2 and Corollary 2.8, we have that N is a normal subgroup having a global fixed point x_0, and that G/N is a finite-rank Abelian group without torsion (so it may be identifies with a subgroup of $\mathbb{Q}^n$). Then, it is easy to see that G/N has a symmetric generating system $\mathcal{G}_0$ such that $\mathcal{G}_0(x_0)$ is bounded. So the corollary follows for $\mathcal{G}$ being the inverse image of $\mathcal{G}_0$ inside G. $\square$

Now, we consider the dynamical realization of $(\Gamma, \preceq)$, and we let $N \subset R$ be the set of elements in R fixing some point. Form Lemma 3.2, N has a global fixed point. It follows from the discussion of §2.3 that N is a normal subgroup of Γ containing $[R, R]$ and that R/N is torsion free. Being normal, N has sequences of global fixed points going to $\pm\infty$. We can therefore define $S = Stab_\Gamma(I)$, where I is the minimal (finite) interval containing 0 whose end points are globally fixed by N. S is a convex subgroup, since for every γ in Γ, either $\gamma(I)$ equals I, or it is disjoint from I.

Case 1: R/N has rank at most 1 and the action of $\widetilde{\Gamma}$ on it is by multiplication by ± 1. In this case we have

Lemma 3.4. *$\widetilde{\Gamma}/N$ is Abelian.*

Proof: All we need to prove is that Γ centralizes R/N since $[\widetilde{\Gamma},\widetilde{\Gamma}]\subseteq R$. So we assume that there exists r in R and $\gamma\in\Gamma$ such that modulo N, $\gamma r\gamma^{-1}=r^{-1}n$, where $n\in N$.

Indeed, since r acts without fixed points, by eventually changing r by r^{-1}, we can assume that $r(y)>y$ for every $y\in\mathbb{R}$. Now, let $x\in\mathbb{R}$ be such that $n(x)=x$ for every $n\in N$. It follows that $\gamma r\gamma^{-1}(x)=r^{-1}n(x)=r^{-1}(x)<x$, which implies that $r\gamma^{-1}(x)<\gamma^{-1}(x)$. This contradiction settles the lemma. □

Proof of case 1: First note that S has smaller Hirsch number that $\widetilde{\Gamma}$. Indeed, $S\cap\widetilde{\Gamma}$ cannot be equal to $\widetilde{\Gamma}$, since the latter does not have global fixed points. On the other hand, since N is contained in $S\cap\widetilde{\Gamma}$, Lemma 3.4 implies that $S\cap\widetilde{\Gamma}$ is a normal subgroup of $\widetilde{\Gamma}$ and that $\widetilde{\Gamma}/S\cap\widetilde{\Gamma}$ is Abelian. So, the quotient $\widetilde{\Gamma}/S\cap\widetilde{\Gamma}$, being left-orderable, has rank >0. It follows that the space of left-orderings of S is either finite or without isolated points. Hence, from Proposition 2.1, we have that $\preceq$ may be isolated only if S is a Tararin group.

However, if S is a Tararin group, we have that every ordering on S is Conradian (see Remark 2.2). This implies, due to the convexity of $S\cap\widetilde{\Gamma}$ in $\widetilde{\Gamma}$ and the fact that $\widetilde{\Gamma}/S\cap\widetilde{\Gamma}$ is Abelian, that the restriction of $\preceq$ to $\widetilde{\Gamma}$ is of Conrad type. Therefore, from the Theorem of Remthulla and Rolfsen in [15], we have that $\preceq$ is a Conradian ordering of Γ. As Γ itself is not Tararin, [10, Proposition 4.1] implies that $\preceq$ is not isolated. □

Case 2: Assume case 1 does not hold. Then, either $\mathrm{rank}(R/N)\geq 2$ or $\mathrm{rank}(R/N)=1$ and there exists $\gamma\in\Gamma$ which does not merely act on R/N by multiplication by ±1. In particular R/N is not isomorphic to $\mathbb{Z}$.

The following may be deduce form [13] (or see [12, Sec. 2.2.5])

Lemma 3.5. *There is a morphism $\varphi:\Gamma\to Aff_+(\mathbb{R})$ such that $\varphi(R)$ contains non-trivial translations, and $N=ker(\varphi)\cap R$. Moreover, this affine action is semi-conjugated to the dynamical realization of* $(\Gamma,\preceq)$.

Sketch of proof: Corollary 3.3 implies that the real line admits an R-invariant measure μ who has no atoms and is finite on compact sets. Moreover since R/N is non-isomorphic to $\mathbb{Z}$, it follows that the translation number morphism $\tau_\mu:R\to\mathbb{R}$ given by $\tau_\mu(g)=\mu([0,g(0)])$ has dense image (here and below, we use the convention $\mu([x,y])=-\mu([y,x])$ when $y<x$), so μ is unique up to scalar multiple. N is precisely the kernel of this morphism. From the uniqueness of μ up to scalar multiple, and the normality of R in Γ, it follows that there is another morphism $\lambda:\Gamma\to\mathbb{R}^*_+$ (here $\mathbb{R}^*_+$ denotes the positives real number under multiplication) $\lambda(g)=\lambda_g$, where λ_g is the ratio $g_*(\mu)/\mu$ (more precisely, $\mu(g^{-1}(A))/\mu(A)$ for a (or any) μ-measurable set A). Then we define

$$\varphi(\gamma)(x)=\frac{1}{\lambda_\gamma}\,x+\mu([0,\gamma(0)]),$$

which extends τ_μ.

To see that this affine action is conjugated to the dynamical realization of $(\Gamma,\preceq)$, we let, for $x\in\mathbb{R}$, $F(x)=\mu([0,x])$. Note that F is non-decreasing. A direct computation shows that

$$F(\gamma(x))=\varphi(\gamma)(F(x)), \tag{1}$$

providing the desired affine action of Γ. □

In what follows we keep the notation of the proof of Lemma 3.5. We let $I_\mu:=(a,b)$, where $a=\sup\{x<0\mid x\in supp(\mu)\}$ and $b=\inf\{x>0\mid x\in supp(\mu)\}$, and $S_\mu=Stab_\Gamma(I_\mu)$. As with S, we have that S_μ is a convex subgroup.

Remark 3.6. Since $\varphi(S_\mu)$ does not contain any non-trivial translation, we deduce that $S_\mu \cap R = N$.

If S_μ had infinitely many orderings, then we could conclude by induction since it has smaller Hirsh number than Γ. So we shall assume that S_μ is a Tararin group. In this case we have

Lemma 3.7. *If S_μ is a Tararin group, then* $\ker \phi$ *is convex.*

Proof: Dynamical inspection shows that $\varphi(S_\mu)$ can only contain homotheties centered at 0, so in particular it is Abelian. If it was trivial, then we would have that $S_\mu = \ker \varphi$. Let us therefore suppose that it is non-trivial.

Now we let $\{id\} = S_0 \lhd S_1 \lhd \ldots \lhd S_n = S_\mu$ be the convex series of the Tararin group S_μ. Recall that S_i/S_{i-1} has rank 1 and that the action of S_{i+1} on S_i/S_{i-1} is by multiplication by some negative number. In particular it has a unique torsion-free Abelian quotient which coincides with $\varphi(S_\mu)$, and therefore $\ker \varphi = S_{n-1}$ is convex. □

Finally, observe that either $\Gamma / \ker \varphi$ is Abelian of rank at least 2, in which case it has no isolated left-orderings [20], or it is a non-Abelian subgroup of the affine group. So we are left with the case of an affine action, which was treated in Lemma 3.1. This finish the proof of Theorem 1.1. □

3.1 An example: the case of SOL_2

Let $\Gamma = \mathbb{Z}^2 \rtimes_T \mathbb{Z}$, where T is an hyperbolic matrix. We denote by H the subgroup $(\mathbb{Z}^2, 0) = [\Gamma, \Gamma]$, and by t the element of Γ acting on H as T. Since T is hyperbolic, we have that H is the nilpotent radical of Γ.

Let $\preceq$ be a left-ordering on Γ, and consider its dynamical realization.

Since T is $\mathbb{Q}$-irreducible, and H is Abelian (thus admitting only Conradian orderings) and finitely generated, we have that there is an element in H having a fixed point, if and only if every element of H has a fixed point, if and only if H has a global fixed point.

Case 1: H has a global fixed point.

In this case, since Γ acts without global fixed points, and H is normal in Γ, we have that, actually, H has infinitely many global fixed points, all permuted by Γ. In particular, S, the stabilizer of the interval given by to consecutive global fixed points of H, is H itself. So H is convex, and $\preceq$ is a Conradian ordering (obviously non-isolated since H is rank-two Abelian).

Case 2: H has no global fixed point.

In this case we have that H is semi-conjugated to a group of translation, thus it preserves a measure without atoms μ. The uniqueness from this measure follows from the fact that the Lebesgue measure is the unique measure, up to scalar multiple, preserved by $\mathbb{Z}^2$ acting faithfully by translations. Finally, the hyperbolicity of T implies that t does not preserve the measure, but it acts on it as a dilation (by the eigenvalues of T). In this way we end up with a faithful embedding of Γ in $Aff_+(\mathbb{R})$, which is realized by a semi-conjugation.

The fact that in this case $\preceq$ is non-isolated follows directly form the proof of Lemma 3.1. □

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